Balkan MO 2006

Nicosia, Cyprus

1 Let $a, b, c > 0$ be real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

Greece

2 Let ABC be a triangle and m a line which intersects the sides AB and AC at interior points D and F, respectively, and intersects the line BC at a point E such that C lies between B and E. The parallel lines from the points A, B, C to the line m intersect the circumcircle of triangle ABC at the points A_1, B_1 and C_1, respectively (apart from A, B, C). Prove that the lines A_1E , B_1F and C_1D pass through the same point.

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3 Find all triplets of positive rational numbers (m, n, p) such that the numbers $m+\frac{1}{np}$, $n+\frac{1}{pm}$, $p+\frac{1}{mn}$ are integers.

Valentin Vornicu, Romania

4 Let m be a positive integer and $\{a_n\}_{n\geq 0}$ be a sequence given by $a_0 = a \in \mathbb{N}$, and

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \equiv 0 \pmod 2, \\ a_n + m & \text{otherwise.} \end{cases}$$

Find all values of a such that the sequence is periodical (starting from the beginning).