Romania
Junior Balkan Team Selection Tests
Timisoara, Bucharest 2008

Day 1

1. Let p be a prime number, $p \neq 3$, and integers a, b such that $p|a+b$ and $p^2|a^3+b^3$. Prove that $p^2 \mid a+b$ or $p^3 \mid a^3+b^3$.

2. Prove that for every $n \in \mathbb{N}^*$ exists a multiple of n, having sum of digits equal to n.

3. Let ABC be an acute-angled triangle. We consider the equilateral triangle $A'UV$, where $A' \in (BC)$, $U \in (AC)$ and $V \in (AB)$ such that $UV \parallel BC$. We define the points B', C' in the same way. Prove that AA', BB' and CC' are concurrent.

4. Let ABC be a triangle, and D the midpoint of the side BC. On the sides AB and AC we consider the points M and N, respectively, both different from the midpoints of the sides, such that
$$AM^2 + AN^2 = BM^2 + CN^2 \text{ and } \angle MDN = \angle BAC.$$
Prove that $\angle BAC = 90^\circ$.

5. Let n be an integer, $n \geq 2$, and the integers $a_1, a_2, \ldots, a_n$, such that $0 < a_k \leq k$, for all $k = 1, 2, \ldots, n$. Knowing that the number $a_1 + a_2 + \cdots + a_n$ is even, prove that there exists a choosing of the signs $+$, respectively $-$, such that
$$a_1 \pm a_2 \pm \cdots \pm a_n = 0.$$

Day 2

1. Consider the acute-angled triangle ABC, altitude AD and point E - intersection of BC with diameter from A of circumcircle. Let M, N be symmetric points of D with respect to the lines AC and AB respectively. Prove that $\angle EMC = \angle BNE$.

2. In a sequence of natural numbers $a_1, a_2, ..., a_n$ every number a_k represents sum of the multiples of the k from sequence. Find all possible values for n.

3. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{1}{a_i^2}.$$

Prove that for every $i = 1, 2, \ldots, n$ we can find i numbers with sum at least i.

4. Let a, b be real nonzero numbers, such that number $\lfloor an + b \rfloor$ is an even integer for every $n \in \mathbb{N}$. Prove that a is an even integer.

Day 3

1. From numbers $1, 2, 3, ..., 37$ we randomly choose 10 numbers. Prove that among these exist four distinct numbers, such that sum of two of them equals to the sum of other two.

2. Let a, b, c be positive reals with $ab + bc + ca = 3$. Prove that:

$$\frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(a + c)} + \frac{1}{1 + c^2(b + a)} \leq \frac{1}{abc}.$$

3. Solve in prime numbers $2p^q - q^p = 7$.

4. Let d be a line and points M, N on the d. Circles $\alpha, \beta, \gamma, \delta$ with centers A, B, C, D are tangent to d, circles α, β are externally tangent at M, and circles γ, δ are externally tangent at N. Points A, C are situated in the same half-plane, determined by d. Prove that if exists an circle, which is tangent to the circles $\alpha, \beta, \gamma, \delta$ and contains them in its interior, then lines AC, BD, MN are concurrent or parallel.

Day 4

1. Let $ABCD$ be a convex quadrilateral with opposite side not parallel. The line through A parallel to BD intersect line CD in F, but parallel through D to AC intersect line AB at E. Denote by M, N, P, Q midpoints of the segments AC, BD, AF, DE. Prove that lines MN, PQ and AD are concurrent.

2. Let m, n be two natural nonzero numbers and sets $A = \{1, 2, ..., n\}, B = \{1, 2, ..., m\}$. We say that subset S of Cartesian product $A \times B$ has property (j) if $(a-x)(b-y) \geq 0$ for each pairs $(a, b), (x, y) \in S$. Prove that every set S with propery (j) has at most $m + n - 1$ elements.

3. Find all pairs (m, n) of integer numbers $m, n > 1$ with property that $mn - 1 \mid n^3 - 1$.

4. Determine the maximum possible real value of the number k, such that

$$(a+b+c)\left(\frac{1}{a+b} + \frac{1}{c+b} + \frac{1}{a+c} - k\right) \geq k$$

for all real numbers $a, b, c \geq 0$ with $a + b + c = ab + bc + ca$.