Romania

Junior Balkan Team Selection Tests

Pitesti and Bucuresti 2007

Day 1 - *13 April 2007*

1. Let us consider a, b two integers. Prove that there exists and it is unique a pair of integers (x, y) such that:
$$(x+2y-a)^2+(2x-y-b)^2 \leq 1.$$

2. Let $ABCD$ be a trapezium ($AB \parallel CD$) and M, N be the intersection points of the circles of diameters AD and BC. Prove that $O \in MN$, where $O \in AC \cap BD$.

3. A rectangularly paper is divided in polygons areas in the following way: at every step one of the existing surfaces is cut by a straight line, obtaining two new areas. Which is the minimum number of cuts needed such that between the obtained polygons there exists 251 polygons with 11 sides?

Day 2 - *14 April 2007*

$\boxed{1}$ Find the positive integers n with $n \geq 4$ such that $[\sqrt{n}]+1$ divides $n-1$ and $[\sqrt{n}]-1$ divides $n+1$.

$\boxed{2}$ Consider a convex quadrilateral $ABCD$. Denote $M,\ N$ the points of tangency of the circle inscribed in $\triangle ABD$ with $AB,\ AD$, respectively and $P,\ Q$ the points of tangency of the circle inscribed in $\triangle CBD$ with the sides $CD,\ CB$, respectively. Assume that the circles inscribed in $\triangle ABD,\ \triangle CBD$ are tangent. Prove that:

a) $ABCD$ is circumscriptible.

b) $MNPQ$ is cyclic.

c) The circles inscribed in $\triangle ABC,\ \triangle ADC$ are tangent.

$\boxed{3}$ Let ABC an isosceles triangle, P a point belonging to its interior. Denote M, N the intersection points of the circle $\mathcal{C}(A, AP)$ with the sides AB and AC, respectively.

Find the position of P if $MN + BP + CP$ is minimum.

Day 3

$\boxed{1}$ Let ABC a triangle and M, N, P points on AB, BC, respective CA, such that the quadrilateral $CPMN$ is a paralelogram. Denote $R \in AN \cap MP$, $S \in BP \cap MN$, and $Q \in AN \cap BP$. Prove that $[MRQS] = [NQP]$.

$\boxed{2}$ Solve in positive integers: $(x^2 + 2)(y^2 + 3)(z^2 + 4) = 60xyz$.

$\boxed{3}$ Consider a nxn table such that the unit squares are colored arbitrary in black and white, such that exactly three of the squares placed in the corners of the table are white, and the other one is black. Prove that there exists a $2x2$ square which contains an odd number of unit squares white colored.

$\boxed{4}$ Let a, b, c three positive reals such that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1.$$

Show that

$$a + b + c \geq ab + bc + ca.$$

Day 4

$\boxed{1}$ Find all nonzero subsets A of the set $\{2, 3, 4, 5, \cdots\}$ such that $\forall n \in A$, we have that $n^2 + 4$ and $\lfloor \sqrt{n} \rfloor + 1$ are both in A.

$\boxed{2}$ Let w_1 and w_2 be two circles which intersect at points A and B. Consider w_3 another circle which cuts w_1 in D, E, and it is tangent to w_2 in the point C, and also tangent to AB in F. Consider $G \in DE \cap AB$, and H the symetric point of F w.r.t G. Find $\angle HCF$.

$\boxed{3}$ Consider the numbers from 1 to 16. The quot;solitarquot; game consists in the arbitrary grouping of the numbers in pairs and replacing each pair with the great prime divisor of the sum of the two numbers (i.e from $(1, 2); (3, 4); (5, 6); ...; (15, 16)$ the numbers which result are $3, 7, 11, 5, 19, 23, 3, 31$). The next step follows from the same procedure and the games continues untill we obtain only one number. Which is the maximum numbers with which the game ends.

$\boxed{4}$ Find all integer positive numbers n such that: $n = [a, b] + [b, c] + [c, a]$, where a, b, c are integer positive numbers and $[p, q]$ represents the least common multiple of numbers p, q.

Day 5

$\boxed{1}$ Consider ρ a semicircle of diameter AB. A parallel to AB cuts the semicircle at C, D such that AD separates B, C. The parallel at AD through C intersects the semicircle the second time at E. Let F be the intersection point of the lines BE and CD. The parallel through F at AD cuts AB in P. Prove that PC is tangent to ρ.

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$\boxed{2}$ Let $x, y, z \geq 0$ be real numbers. Prove that:

$$\frac{x^3 + y^3 + z^3}{3} \geq xyz + \frac{3}{4}|(x - y)(y - z)(z - x)|.$$

Additional task: Find the maximal real constant α that can replace $\frac{3}{4}$ such that the inequality is still true for any non-negative x, y, z.

$\boxed{3}$ At a party there are eight guests, and each participant can't talk with at most three persons. Prove that we can group the persons in four pairs such that in every pair a conversation can take place.

$\boxed{4}$ We call a set of points *free* if there is no equilateral triangle with the vertices among the points of the set. Prove that every set of n points in the plane contains a *free* subset with at least $\sqrt{n}$ elements.

Day 6

1. Consider an 8x8 board divided in 64 unit squares. We call *diagonal* in this board a set of 8 squares with the property that on each of the rows and the columns of the board there is exactly one square of the *diagonal*. Some of the squares of this board are coloured such that in every *diagonal* there are exactly two coloured squares. Prove that there exist two rows or two columns whose squares are all coloured.

2. There are given the integers $1 \leq m < n$. Consider the set $M = \{(x, y); x, y \in \mathbb{Z}_+, 1 \leq x, y \leq n\}$. Determine the least value $v(m, n)$ with the property that for every subset $P \subseteq M$ with $|P| = v(m, n)$ there exist $m+1$ elements $A_i = (x_i, y_i) \in P, i = 1, 2, ..., m+1$, for which the x_i are all distinct, and y_i are also all distinct.

3. Let ABC be a right triangle with $A = 90^\circ$ and $D \in (AC)$. Denote by E the reflection of A in the line BD and F the intersection point of CE with the perpendicular in D to BC. Prove that AF, DE and BC are concurrent.

4. We call a real number x with $0 < x < 1$ *interesting* if x is irrational and if in its decimal writing the first four decimals are equal. Determine the least positive integer n with the property that every real number t with $0 < t < 1$ can be written as the sum of n pairwise distinct interesting numbers.

