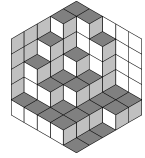




Romania
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Day 1 - 19 April 2006

- 1] Let ABC and AMN be two similar triangles with the same orientation, such that $AB = AC$, $AM = AN$ and having disjoint interiors. Let O be the circumcenter of the triangle MAB . Prove that the points O, C, N, A lie on the same circle if and only if the triangle ABC is equilateral.

Valentin Vornicu

- 2] Let p a prime number, $p \geq 5$. Find the number of polynomials of the form

$$x^p + px^k + px^l + 1, \quad k > l, \quad k, l \in \{1, 2, \dots, p-1\},$$

which are irreducible in $\mathbb{Z}[X]$.

Valentin Vornicu

- 4] The real numbers a_1, a_2, \dots, a_n are given such that $|a_i| \leq 1$ for all $i = 1, 2, \dots, n$ and $a_1 + a_2 + \dots + a_n = 0$.

a) Prove that there exists $k \in \{1, 2, \dots, n\}$ such that

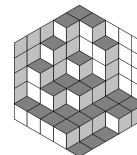
$$|a_1 + 2a_2 + \dots + ka_k| \leq \frac{2k+1}{4}.$$

b) Prove that for $n > 2$ the bound above is the best possible.

Radu Gologan, Calin Popescu



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Day 2 - 20 April 2006

- 1] Let $\{a_n\}_{n \geq 1}$ be a sequence with $a_1 = 1$, $a_2 = 4$ and for all $n > 1$,

$$a_n = \sqrt{a_{n-1}a_{n+1} + 1}.$$

- a) Prove that all the terms of the sequence are positive integers.
b) Prove that $2a_n a_{n+1} + 1$ is a perfect square for all positive integers n .

Valentin Vornicu

- 2] Let ABC be a triangle with $\angle B = 30^\circ$. We consider the closed disks of radius $\frac{AC}{3}$, centered in A, B, C . Does there exist an equilateral triangle with one vertex in each of the 3 disks?

Radu Gologan

- 3] For which pairs of positive integers (m, n) there exists a set A such that for all positive integers x, y , if $|x - y| = m$, then at least one of the numbers x, y belongs to the set A , and if $|x - y| = n$, then at least one of the numbers x, y does not belong to the set A ?

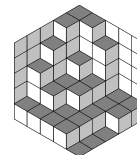
A.M.M.

- 4] Let x_i , $1 \leq i \leq n$ be real numbers. Prove that

$$\sum_{1 \leq i < j \leq n} |x_i + x_j| \geq \frac{n-2}{2} \sum_{i=1}^n |x_i|.$$



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Day 3 - 16 May 2006

- 1] The circle of center I is inscribed in the convex quadrilateral $ABCD$. Let M and N be points on the segments AI and CI , respectively, such that $\angle MBN = \frac{1}{2}\angle ABC$. Prove that $\angle MDN = \frac{1}{2}\angle ADC$.
- 2] Let A be point in the exterior of the circle \mathcal{C} . Two lines passing through A intersect the circle \mathcal{C} in points B and C (with B between A and C) respectively in D and E (with D between A and E). The parallel from D to BC intersects the second time the circle \mathcal{C} in F . Let G be the second point of intersection between the circle \mathcal{C} and the line AF and M the point in which the lines AB and EG intersect. Prove that

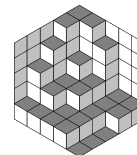
$$\frac{1}{AM} = \frac{1}{AB} + \frac{1}{AC}.$$

- 3] Let γ be the incircle in the triangle $A_0A_1A_2$. For all $i \in \{0, 1, 2\}$ we make the following constructions (all indices are considered modulo 3): γ_i is the circle tangent to γ which passes through the points A_{i+1} and A_{i+2} ; T_i is the point of tangency between γ_i and γ ; finally, the common tangent in T_i of γ_i and γ intersects the line $A_{i+1}A_{i+2}$ in the point P_i . Prove that
- a) the points P_0 , P_1 and P_2 are collinear;
 - b) the lines A_0T_0 , A_1T_1 and A_2T_2 are concurrent.
- 4] Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2.$$



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Day 4 - 19 May 2006

- 1] Let r and s be two rational numbers. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that for all $x, y \in \mathbb{Q}$ we have

$$f(x + f(y)) = f(x + r) + y + s.$$

- 2] Find all non-negative integers m, n, p, q such that

$$p^m q^n = (p + q)^2 + 1.$$

- 3] Let $n > 1$ be an integer. A set $S \subset \{0, 1, 2, \dots, 4n - 1\}$ is called *rare* if, for any $k \in \{0, 1, \dots, n - 1\}$, the following two conditions take place at the same time

(1) the set $S \cap \{4k - 2, 4k - 1, 4k, 4k + 1, 4k + 2\}$ has at most two elements;

(2) the set $S \cap \{4k + 1, 4k + 2, 4k + 3\}$ has at most one element.

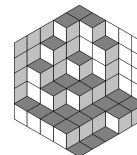
Prove that the set $\{0, 1, 2, \dots, 4n - 1\}$ has exactly $8 \cdot 7^{n-1}$ rare subsets.

- 4] Let p, q be two integers, $q \geq p \geq 0$. Let $n \geq 2$ be an integer and $a_0 = 0, a_1 \geq 0, a_2, \dots, a_{n-1}, a_n = 1$ be real numbers such that

$$a_k \leq \frac{a_{k-1} + a_{k+1}}{2}, \quad \forall k = 1, 2, \dots, n - 1.$$

Prove that

$$(p + 1) \sum_{k=1}^{n-1} a_k^p \geq (q + 1) \sum_{k=1}^{n-1} a_k^q.$$



Day 5 - 20 May 2006

- 1] Let n be a positive integer of the form $4k + 1$, $k \in \mathbb{N}$ and $A = \{a^2 + nb^2 \mid a, b \in \mathbb{Z}\}$. Prove that there exist integers x, y such that $x^n + y^n \in A$ and $x + y \notin A$.
- 2] Let m and n be positive integers and S be a subset with $(2^m - 1)n + 1$ elements of the set $\{1, 2, 3, \dots, 2^m n\}$. Prove that S contains $m + 1$ distinct numbers a_0, a_1, \dots, a_m such that $a_{k-1} \mid a_k$ for all $k = 1, 2, \dots, m$.
- 3] Let $x_1 = 1, x_2, x_3, \dots$ be a sequence of real numbers such that for all $n \geq 1$ we have

$$x_{n+1} = x_n + \frac{1}{2x_n}.$$

Prove that

$$[25x_{625}] = 625.$$

- 4] Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A and ω the circumcircle of the triangle. Let ω_1 be the circle tangent to AD, BD and ω . Let ω_2 be the circle tangent to AD, CD and ω . Let ℓ be the interior common tangent to both ω_1 and ω_2 , different from CD .

Prove that ℓ passes through the midpoint of BC if and only if $2BC = AB + AC$.