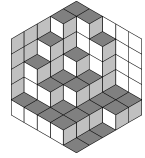




**Romania**  
**Team Selection Tests**  
2005



**Day 1 - 31 March 2005**

- 1 Solve the equation  $3^x = 2^x y + 1$  in positive integers.
- 2 Let  $n \geq 1$  be an integer and let  $X$  be a set of  $n^2 + 1$  positive integers such that in any subset of  $X$  with  $n + 1$  elements there exist two elements  $x \neq y$  such that  $x \mid y$ . Prove that there exists a subset  $\{x_1, x_2, \dots, x_{n+1}\} \in X$  such that  $x_i \mid x_{i+1}$  for all  $i = 1, 2, \dots, n$ .
- 3 Prove that if the distance from a point inside a convex polyhedra with  $n$  faces to the vertices of the polyhedra is at most 1, then the sum of the distances from this point to the faces of the polyhedra is smaller than  $n - 2$ .

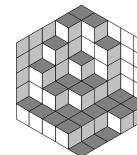
*Calin Popescu*



# Romania

## Team Selection Tests

2005



### Day 2 - 01 April 2005

- 1] Prove that in any convex polygon with  $4n + 2$  sides ( $n \geq 1$ ) there exist two consecutive sides which form a triangle of area at most  $\frac{1}{6n}$  of the area of the polygon.
- 2] Let  $m, n$  be co-prime integers, such that  $m$  is even and  $n$  is odd. Prove that the following expression does not depend on the values of  $m$  and  $n$ :

$$\frac{1}{2n} + \sum_{k=1}^{n-1} (-1)^{\lfloor \frac{mk}{n} \rfloor} \left\{ \frac{mk}{n} \right\}.$$

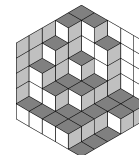
*Bogdan Enescu*

- 3] A sequence of real numbers  $\{a_n\}_n$  is called a *bs* sequence if  $a_n = |a_{n+1} - a_{n+2}|$ , for all  $n \geq 0$ . Prove that a *bs* sequence is bounded if and only if the function  $f$  given by  $f(n, k) = a_n a_k (a_n - a_k)$ , for all  $n, k \geq 0$  is the null function.

*Mihai Baluna - ISL 2004*



**Romania**  
**Team Selection Tests**  
2005



**Day 3 - 19 April 2005**

- 1] Let  $A_0A_1A_2A_3A_4A_5$  be a convex hexagon inscribed in a circle. Define the points  $A'_0, A'_2, A'_4$  on the circle, such that

$$A_0A'_0 \parallel A_2A_4, \quad A_2A'_2 \parallel A_4A_0, \quad A_4A'_4 \parallel A_0A_2.$$

Let the lines  $A'_0A_3$  and  $A_2A_4$  intersect in  $A'_3$ , the lines  $A'_2A_5$  and  $A_0A_4$  intersect in  $A'_5$  and the lines  $A'_4A_1$  and  $A_0A_2$  intersect in  $A'_1$ .

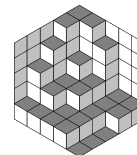
Prove that if the lines  $A_0A_3, A_1A_4$  and  $A_2A_5$  are concurrent then the lines  $A_0A'_3, A_4A'_1$  and  $A_2A'_5$  are also concurrent.

- 2] Let  $ABC$  be a triangle, and let  $D, E, F$  be 3 points on the sides  $BC, CA$  and  $AB$  respectively, such that the inradii of the triangles  $AEF, BDF$  and  $CDE$  are equal with half of the inradius of the triangle  $ABC$ . Prove that  $D, E, F$  are the midpoints of the sides of the triangle  $ABC$ .

- 3] Let  $P$  be a polygon (not necessarily convex) with  $n$  vertices, such that all its sides and diagonals are less or equal with 1 in length. Prove that the area of the polygon is less than  $\frac{\sqrt{3}}{2}$ .



**Romania**  
**Team Selection Tests**  
2005



**Day 4 - 23 May 2005**

- 1] Let  $a \in \mathbb{R} - \{0\}$ . Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(a + x) = f(x) - x$  for all  $x \in \mathbb{R}$ .

*Dan Schwartz*

- 2] On the edges of a convex polyhedra we draw arrows such that from each vertex at least an arrow is pointing in and at least one is pointing out. Prove that there exists a face of the polyhedra such that the arrows on its edges form a circuit.

*Dan Schwartz*

- 3] Let  $n \geq 0$  be an integer and let  $p \equiv 7 \pmod{8}$  be a prime number. Prove that

$$\sum_{k=1}^{p-1} \left\{ \frac{k^{2^n}}{p} - \frac{1}{2} \right\} = \frac{p-1}{2}.$$

*Clin Popescu*

- 4] a) Prove that there exists a sequence of digits  $\{c_n\}_{n \geq 1}$  such that for each  $n \geq 1$  no matter how we interlace  $k_n$  digits,  $1 \leq k_n \leq 9$ , between  $c_n$  and  $c_{n+1}$ , the infinite sequence thus obtained does not represent the fractional part of a rational number.

b) Prove that for  $1 \leq k_n \leq 10$  there is no such sequence  $\{c_n\}_{n \geq 1}$ .

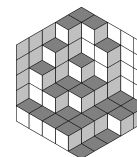
*Dan Schwartz*



# Romania

## Team Selection Tests

2005



### Day 5 - 24 May 2005

- 1 On a  $2004 \times 2004$  chess table there are 2004 queens such that no two are attacking each other<sup>1</sup>.

Prove that there exist two queens such that in the rectangle in which the center of the squares on which the queens lie are two opposite corners, has a semiperimeter of 2004.

- 2 Let  $n \geq 2$  be an integer. Find the smallest real value  $\rho(n)$  such that for any  $x_i > 0$ ,  $i = 1, 2, \dots, n$  with  $x_1 x_2 \cdots x_n = 1$ , the inequality

$$\sum_{i=1}^n \frac{1}{x_i} \leq \sum_{i=1}^n x_i^r$$

is true for all  $r \geq \rho(n)$ .

- 3 Let  $\mathbb{N} = \{1, 2, \dots\}$ . Find all functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $m, n \in \mathbb{N}$  the number  $f^2(m) + f(n)$  is a divisor of  $(m^2 + n)^2$ .

- 4 We consider a polyhedra which has exactly two vertices adjacent with an odd number of edges, and these two vertices are lying on the same edge.

Prove that for all integers  $n \geq 3$  there exists a face of the polyhedra with a number of sides not divisible by  $n$ .

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<sup>1</sup>two queens attack each other if they lie on the same row, column or direction parallel with on of the main diagonals of the table