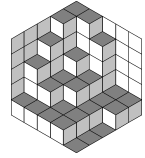




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Day 1

- 1] Let a_1, a_2, a_3, a_4 be the sides of an arbitrary quadrilateral of perimeter $2s$. Prove that

$$\sum_{i=1}^4 \frac{1}{a_i + s} \leq \frac{2}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{(s - a_i)(s - a_j)}}.$$

When does the equality hold?

- 2] Let $\{R_i\}_{1 \leq i \leq n}$ be a family of disjoint closed rectangular surfaces with total area 4 such that their projections of the Ox axis is an interval. Prove that there exist a triangle with vertices in $\bigcup_{i=1}^n R_i$ which has an area of at least 1.

[Thanks Grobber for the correction]

- 3] Find all one-to-one mappings $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all positive integers n the following relation holds:

$$f(f(n)) \leq \frac{n + f(n)}{2}.$$

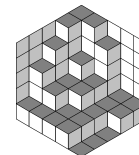
- 4] Let D be a closed disc in the complex plane. Prove that for all positive integers n , and for all complex numbers $z_1, z_2, \dots, z_n \in D$ there exists a $z \in D$ such that $z^n = z_1 \cdot z_2 \cdots z_n$.



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Day 2

- 5] A circular disk is partitioned into $2n$ equal sectors by n straight lines through its center. Then, these $2n$ sectors are colored in such a way that exactly n of the sectors are colored in blue, and the other n sectors are colored in red. We number the red sectors with numbers from 1 to n in counter-clockwise direction (starting at some of these red sectors), and then we number the blue sectors with numbers from 1 to n in clockwise direction (starting at some of these blue sectors).

Prove that one can find a half-disk which contains sectors numbered with all the numbers from 1 to n (in some order). (In other words, prove that one can find n consecutive sectors which are numbered by all numbers 1, 2, ..., n in some order.)

Problem 8 from CWMO 2007: n white and n black balls are placed at random on the circumference of a circle. Starting from a certain white ball, number all white balls in a clockwise direction by 1, 2, ..., n . Likewise number all black balls by 1, 2, ..., n in anti-clockwise direction starting from a certain black ball. Prove that there exists a chain of n balls whose collection of numbering forms the set $\{1, 2, 3, \dots, n\}$.

- 6] Let a, b be two positive integers, such that $ab \neq 1$. Find all the integer values that $f(a, b)$ can take, where

$$f(a, b) = \frac{a^2 + ab + b^2}{ab - 1}.$$

- 7] Let a, b, c be 3 integers, b odd, and define the sequence $\{x_n\}_{n \geq 0}$ by $x_0 = 4$, $x_1 = 0$, $x_2 = 2c$, $x_3 = 3b$ and for all positive integers n we have

$$x_{n+3} = ax_{n-1} + bx_n + cx_{n+1}.$$

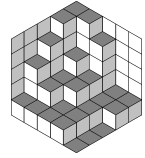
Prove that for all positive integers m , and for all primes p the number x_{p^m} is divisible by p .

- 8] Let Γ be a circle, and let $ABCD$ be a square lying inside the circle Γ . Let \mathcal{C}_a be a circle tangent interiorly to Γ , and also tangent to the sides AB and AD of the square, and also lying inside the opposite angle of $\angle BAD$. Let A' be the tangency point of the two circles. Define similarly the circles $\mathcal{C}_b, \mathcal{C}_c, \mathcal{C}_d$ and the points B', C', D' respectively.

Prove that the lines AA', BB', CC' and DD' are concurrent.



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Day 3

- 9] Let $n \geq 2$ be a positive integer, and X a set with n elements. Let A_1, A_2, \dots, A_{101} be subsets of X such that the union of any 50 of them has more than $\frac{50}{51}n$ elements.

Prove that among these 101 subsets there exist 3 subsets such that any two of them have a common element.

- 10] Prove that for all positive integers n, m , with m odd, the following number is an integer

$$\frac{1}{3^m n} \sum_{k=0}^m \binom{3m}{3k} (3n-1)^k.$$

- 11] Let I be the incenter of the non-isosceles triangle ABC and let A', B', C' be the tangency points of the incircle with the sides BC, CA, AB respectively. The lines AA' and BB' intersect in P , the lines AC and $A'C'$ in M and the lines $B'C'$ and BC intersect in N . Prove that the lines IP and MN are perpendicular.

Alternative formulation. The incircle of a non-isosceles triangle ABC has center I and touches the sides BC, CA and AB in A', B' and C' , respectively. The lines AA' and BB' intersect in P , the lines AC and $A'C'$ intersect in M , and the lines BC and $B'C'$ intersect in N . Prove that the lines IP and MN are perpendicular.

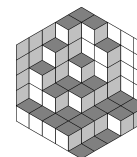
- 12] Let $n \geq 2$ be an integer and let a_1, a_2, \dots, a_n be real numbers. Prove that for any non-empty subset $S \subset \{1, 2, 3, \dots, n\}$ we have

$$\left(\sum_{i \in S} a_i \right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + \dots + a_j)^2.$$

Gabriel Dospinescu



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Day 4

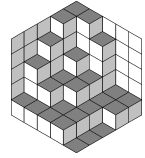
- 13] Let $m \geq 2$ be an integer. A positive integer n has the property that for any positive integer a which is co-prime with n , we have $a^m - 1 \equiv 0 \pmod{n}$.
Prove that $n \leq 4m(2^m - 1)$.
Created by Harazi, modified by Marian Andronache.
- 14] Let O be a point in the plane of the triangle ABC . A circle \mathcal{C} which passes through O intersects the second time the lines OA, OB, OC in P, Q, R respectively. The circle \mathcal{C} also intersects for the second time the circumcircles of the triangles BOC, COA and AOB respectively in K, L, M .
Prove that the lines PK, QL and RM are concurrent.
- 15] Some of the n faces of a polyhedron are colored in black such that any two black-colored faces have no common vertex. The rest of the faces of the polyhedron are colored in white.
Prove that the number of common sides of two white-colored faces of the polyhedron is at least $n - 2$.



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Day 5

- 16] Three circles $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ of radii R_1, R_2, R_3 respectively, pass through the point O and intersect two by two in A, B, C . The point O lies inside the triangle ABC .

Let A_1, B_1, C_1 be the intersection points of the lines AO, BO, CO with the sides BC, CA, AB of the triangle ABC . Let $\alpha = \frac{OA_1}{AA_1}, \beta = \frac{OB_1}{BB_1}$ and $\gamma = \frac{OC_1}{CC_1}$ and let R be the circumradius of the triangle ABC . Prove that

$$\alpha R_1 + \beta R_2 + \gamma R_3 \geq R.$$

- 17] On a chess table $n \times m$ we call a *move* the following succession of operations

(i) choosing some unmarked squares, any two not lying on the same row or column; (ii) marking them with 1; (iii) marking with 0 all the unmarked squares which lie on the same line and column with a square marked with the number 1 (even if the square has been marked with 1 on another move).

We call a *game* a succession of moves that end in the moment that we cannot make any more moves.

What is the maximum possible sum of the numbers on the table at the end of a game?

- 18] Let p be a prime number and $f \in \mathbb{Z}[X]$ given by

$$f(x) = a_{p-1}x^{p-2} + a_{p-2}x^{p-3} + \cdots + a_2x + a_1,$$

where $a_i = \left(\frac{i}{p}\right)$ is the Legendre symbol of i with respect to p (i.e. $a_i = 1$ if $i^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ and $a_i = -1$ otherwise, for all $i = 1, 2, \dots, p-1$).

a) Prove that $f(x)$ is divisible with $(x-1)$, but not with $(x-1)^2$ iff $p \equiv 3 \pmod{4}$; b) Prove that if $p \equiv 5 \pmod{8}$ then $f(x)$ is divisible with $(x-1)^2$ but not with $(x-1)^3$.

Suggested by Calin Popescu.