Romania
Team Selection Tests
2003

Day 1 - *23 April 2003*

$\boxed{1}$ Let $(a_n)_{n\geq 1}$ be a sequence for real numbers given by $a_1 = 1/2$ and for each positive integer n

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}.$$

Prove that for every positive integer n we have $a_1 + a_2 + \cdots + a_n < 1$.

$\boxed{2}$ Let ABC be a triangle with $\angle BAC = 60^\circ$. Consider a point P inside the triangle having $PA = 1$, $PB = 2$ and $PC = 3$. Find the maximum possible area of the triangle ABC.

$\boxed{3}$ Let n, k be positive integers such that $n^k > (k+1)!$ and consider the set

$$M = \{(x_1, x_2, \ldots, x_n) x_i \in \{1, 2, \ldots, n\},\ i = \overline{1, k}\}.$$

Prove that if $A \subset M$ has $(k+1)! + 1$ elements, then there are two elements $\{\alpha, \beta\} \subset A$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ such that

$$(k+1)! \,|\, (\beta_1 - \alpha_1)(\beta_2 - \alpha_2) \cdots (\beta_k - \alpha_k)\ .$$

Day 2 - *24 April 2003*

$\boxed{4}$ Prove that among the elements of the sequence $\left\{\left\lfloor n\sqrt{2003}\right\rfloor\right\}_{n\geq 1}$ one can find a geometric progression having any number of terms, and having the ratio bigger than k, where k can be any positive integer.

Radu Gologan

$\boxed{5}$ Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

Mihai Piticari

$\boxed{6}$ At a math contest there are $2n$ students participating. Each of them submits a problem to the jury, which thereafter gives each students one of the $2n$ problems submitted. One says that the contest is *fair* is there are n participants which receive their problems from the other n participants.

Prove that the number of distributions of the problems in order to obtain a fair contest is a perfect square.

Day 3 - *24 May 2003*

$\boxed{7}$ Find all integers a, b, m, n, with $m > n > 1$, for which the polynomial $f(X) = X^n + aX + b$ divides the polynomial $g(X) = X^m + aX + b$.

Laurentiu Panaitopol

$\boxed{8}$ Two circles ω_1 and ω_2 with radii r_1 and r_2, $r_2 > r_1$, are externally tangent. The line t_1 is tangent to the circles ω_1 and ω_2 at points A and D respectively. The parallel line t_2 to the line t_1 is tangent to the circle ω_1 and intersects the circle ω_2 at points E and F. The line t_3 passing through D intersects the line t_2 and the circle ω_2 in B and C respectively, both different of E and F respectively. Prove that the circumcircle of the triangle ABC is tangent to the line t_1.

Dinu Serbanescu

$\boxed{9}$ Let $n \geq 3$ be a positive integer. Inside a $n \times n$ array there are placed n^2 positive numbers with sum n^3. Prove that we can find a square 2×2 of 4 elements of the array, having the sides parallel with the sides of the array, and for which the sum of the elements in the square is greater than $3n$.

Radu Gologan

Day 4 - *25 May 2003*

10 Let $\mathcal{P}$ the set of all the primes and let M be a subset of $\mathcal{P}$, having at least three elements, and such that for any proper subset A of M all of the prime factors of the number $-1+\prod_{p\in A} p$ are found in M. Prove that $M=\mathcal{P}$.

Valentin Vornicu

11 In a square of side 6 the points A, B, C, D are given such that the distance between any two of the four points is at least 5. Prove that A, B, C, D form a convex quadrilateral and its area is greater than 21.

Laurentiu Panaitopol

12 A word is a sequence of n letters of the alphabet a, b, c, d. A word is said to be complicated if it contains two consecutive groups of identic letters. The words caab, baba and cababdc, for example, are complicated words, while bacba and dcbdc are not. A word that is not complicated is a simple word. Prove that the numbers of simple words with n letters is greater than 2^n, if n is a positive integer.

Day 5 - *19 June 2003*

$\boxed{13}$ A parliament has n senators. The senators form 10 parties and 10 committees, such that any senator belongs to exactly one party and one committee. Find the least possible n for which it is possible to label the parties and the committees with numbers from 1 to 10, such that there are at least 11 senators for which the numbers of the corresponding party and committee are equal.

$\boxed{14}$ Given is a rhombus $ABCD$ of side 1. On the sides BC and CD we are given the points M and N respectively, such that $MC + CN + MN = 2$ and $2\angle MAN = \angle BAD$. Find the measures of the angles of the rhombus.

Cristinel Mortici

$\boxed{15}$ In a plane we choose a cartesian system of coordinates. A point $A(x, y)$ in the plane is called an integer point if and only if both x and y are integers. An integer point A is called invisible if on the segment (OA) there is at least one integer point. Prove that for each positive integer n there exists a square of side n in which all the interior integer points are invisible.

Day 6 - *20 June 2003*

$\boxed{16}$ Let $ABCDEF$ be a convex hexagon and denote by A', B', C', D', E', F' the middle points of the sides AB, BC, CD, DE, EF and FA respectively. Given are the areas of the triangles ABC', BCD', CDE', DEF', EFA' and FAB'. Find the area of the hexagon.

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$\boxed{17}$ A permutation $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called *straight* if and only if for each integer k, $1 \le k \le n - 1$ the following inequality is fulfilled

$$|\sigma(k) - \sigma(k+1)| \le 2.$$

Find the smallest positive integer n for which there exist at least 2003 straight permutations.

Valentin Vornicu

$\boxed{18}$ For every positive integer n we denote by $d(n)$ the sum of its digits in the decimal representation. Prove that for each positive integer k there exists a positive integer m such that the equation $x + d(x) = m$ has exactly k solutions in the set of positive integers.