REPUBLIC OF SAKHA (YAKUTIA)
MINISTRY OF EDUCATION AND SCIENCE

INTERNATIONAL OLYMPIAD "TUYMAADA-2023" (mathematics)

First day

Yakutsk 2023

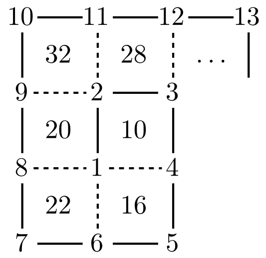
The booklet contains the problems of XXX International school students olympiad "Tuymaada" in mathematics.

The problems were prepared with the participation of members of the Methodical commission of Russian mathematical olympiad. The booklet was compiled by A. V. Antropov, S. L. Berlov, A. S. Golovanov, K. P. Kokhas. Computer typesetting: M. A. Ivanov, K. P. Kokhas, A. I. Khabrov.

Each problem is worth 7 points. Duration of each of the days of the olympiad is 5 hours.

Senior league

1. The numbers 1, 2, 3, ... are arranged in a spiral in the vertices of an infinite square grid (see figure). Then in the centre of each square the sum of the numbers in its vertices is placed. Prove that for each positive integer n the centres of the squares contain infinitely many multiples of n.



(*K. Kokhas*)

2. In a graph with n vertices every two vertices are connected by a unique path. For each two vertices u and v, let $d(u, v)$ denote the distance between u and v, i.e. the number of edges in the path connecting these two vertices, and $\deg u$ denote the degree of a vertex u. Let W be the sum of pairwise distances between the vertices, and D the sum of weighted pairwise distances:

$$D = \sum_{\{u,v\}} (\deg u + \deg v) d(u, v).$$

Prove that $D = 4W - n(n - 1)$.

(*I. Gutman*)

3. Prove for integral $n \geqslant 2$ the inequality

$$\frac{\sqrt[3]{\frac{1}{n+1}} + \sqrt[3]{\frac{2}{n+1}} + \ldots + \sqrt[3]{\frac{n}{n+1}}}{n} \leqslant \frac{\sqrt[3]{\frac{1}{n}} + \sqrt[3]{\frac{2}{n}} + \ldots + \sqrt[3]{\frac{n-1}{n}}}{n-1}.$$

(*J. Liu*)

4. Two points A and B and line ℓ are fixed in the plane so that ℓ is not perpendicular to AB and does not intersect the segment AB. We consider all circles with a centre $O \notin \ell$ passing through A and B and meeting ℓ at some points C and D. Prove that all the circumcircles of triangles OCD touch a fixed circle.

(*S. Berlov*)

Junior League

1. Prove that for a, b, $c \in [0, 1]$ the following inequality holds:

$$(1-a)(1+ab)(1+ac)(1-abc) \leqslant (1+a)(1-ab)(1-ac)(1+abc).$$

(*G. Raposo*)

2. Serge and Tanya want to show Masha a magic trick. Serge leaves the room. Masha writes down a sequence $(a_1, a_2, \ldots, a_n)$, where all a_k equal 0 or 1. After that Tanya writes down a sequence $(b_1, b_2, \ldots, b_n)$, where all b_k also equal 0 or 1. Then Masha either does nothing or says "Mutabor" and replaces both sequences: her own sequence by $(a_n, a_{n-1}, \ldots, a_1)$, and Tanya's sequence by $(1-b_n, 1-b_{n-1}, \ldots, 1-b_1)$. Masha's sequence is covered by a napkin, and Serge is invited to the room. Serge should look at Tanya's sequence and tell the sequence covered by the napkin. For what n Serge and Tanya can prepare and show such a trick? Serge does not have to determine whether the word "Mutabor" has been pronounced.

(*A. Antropov, T. Gizatullin*)

3. Point L inside triangle ABC is such that $CL = AB$ and $\angle BAC + \angle BLC = 180^\circ$. Point K on the side AC is such that $KL \parallel BC$. Prove that $AB = BK$.

(*A. Antropov*)

4. Two players play a game. They have $n > 2$ piles containing $n^{10} + 1$ stones each. A move consists of removing all the piles but one and dividing the remaining pile into n nonempty piles. The player that cannot move loses. Who has a winning strategy, the player that moves first or his adversary?

(*T. Abuku, K. Sakai, M. Shinoda, K. Suetsugu*)