Olimpiada de matematică – clasa a XI-a
etapa zonală – 9 februarie 2013

SOLUȚII

1. Calculați limita $\lim\limits_{n\to\infty}\dfrac{\left[n^2\sqrt{2}+n\sqrt{3}\right]}{\left[n^2\sqrt{3}+n\sqrt{2}\right]}$, unde $[x]$ reprezintă partea întreagă a numărului real x.

Rezolvare

$$\frac{n^2\sqrt{2}+n\sqrt{3}-1}{n^2\sqrt{3}+n\sqrt{2}+1}<\frac{\left[n^2\sqrt{2}+n\sqrt{3}\right]}{\left[n^2\sqrt{3}+n\sqrt{2}\right]}<\frac{n^2\sqrt{2}+n\sqrt{3}+1}{n^2\sqrt{3}+n\sqrt{2}-1}.$$

$$\lim_{n\to\infty}\frac{n^2\sqrt{2}+n\sqrt{3}-1}{n^2\sqrt{3}+n\sqrt{2}+1}=\frac{n^2\sqrt{2}+n\sqrt{3}+1}{n^2\sqrt{3}+n\sqrt{2}-1}=\sqrt{\frac{2}{3}}$$

Deci $\lim\limits_{n\to\infty}\dfrac{\left[n^2\sqrt{2}+n\sqrt{3}\right]}{\left[n^2\sqrt{3}+n\sqrt{2}\right]}=\sqrt{\dfrac{2}{3}}$

2. Fie $z, v \in \mathbb{C}$, $z \neq 0$ două numere complexe, și matricea $X=\begin{pmatrix} z & v \\ 0 & z \end{pmatrix}\in M_2(\mathbb{C})$.

a) Determinați X^n pentru $n\in\mathbb{N}^*$.

b) Determinați $z, v\in\mathbb{C}$ pentru care $X^n=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, $n\in\mathbb{N}^*$, $n\geq 2$.

Rezolvare

Se demonstrează prin inducție matematică (sau folosind Cayley-Hamilton), că

$$X^n=\begin{pmatrix} z^n & nz^{n-1}v \\ 0 & z^n \end{pmatrix}.$$

Din $X^n=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ obținem $z^n=1$, $z^{n-1}v=1 \Rightarrow z=v$ și ambele sunt rădăcini de ordinul n al unității.

3. Fie şirul $(a_n)_{n\geq 1}$, $a_n = \begin{vmatrix} a+x & a & \cdots & a \\ a & a+x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+x \end{vmatrix}$. Calculaţi limita $\lim\limits_{n\to\infty}\left(\lim\limits_{x\to 0}\frac{a_{n+1}}{x\cdot a_n}\right)^n$.

Rezolvare

$$a_n = \begin{vmatrix} na+x & na+x & \cdots & na+x \\ a & a+x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & x & \cdots & a+x \end{vmatrix} = (na+x)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+x & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+x \end{vmatrix} \overset{C_i - C_1}{=}$$

$$(na+x)\begin{vmatrix} 1 & 0 & \cdots & 0 \\ a & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a & 0 & \cdots & x \end{vmatrix} = (na+x)x^{n-1}$$

$$\lim_{x\to 0}\frac{a_{n+1}}{x\cdot a_n} = \lim_{x\to 0}\frac{((n+1)a+x)x^n}{x(na+x)x^{n-1}} = \frac{n+1}{n}$$

$$\lim_{n\to\infty}\left(\lim_{x\to 0}\frac{a_{n+1}}{x\cdot a_n}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

4. Fie şirul de numere reale $(a_n)_{n\geq 1}$ definit de relaţia $2a_{n+1} = a_n + \frac{a_n}{n}$ pentru orice $n \in \mathbb{N}^*$ şi $a_1 = \frac{1}{2}$.

a) Determinaţi termenul general al şirului $(a_n)_{n\geq 1}$

b) Calculaţi limita $\lim\limits_{n\to\infty}\frac{\sqrt[n]{a_1 a_2 \ldots a_n}}{2^{\frac{n+1}{2}}\cdot a_{n+1}}$

Rezolvare

a) $2a_{n+1} = a_n + \frac{a_n}{n} \Leftrightarrow a_{n+1} = a_n\cdot\frac{n+1}{2n} = a_{n-1}\cdot\frac{n}{2(n-1)}\cdot\frac{n+1}{2n} = a_{n-2}\frac{n-1}{2(n-2)}\cdot\frac{n}{2(n-1)}\cdot\frac{n+1}{2n} = \ldots$

$$\ldots = a_1\frac{2}{2\cdot 1}\cdot\frac{3}{2\cdot 2}\cdot\ldots\cdot\frac{n}{2(n-1)}\cdot\frac{n+1}{2n} = \frac{1}{2}\cdot\frac{n+1}{2^n} = \frac{n+1}{2^{n+1}} \Rightarrow a_n = \frac{n}{2^n}$$

b) $\dfrac{\sqrt[n]{a_1 a_2 \ldots a_n}}{2^{\frac{n+1}{2}}\cdot a_{n+1}} = \dfrac{\sqrt[n]{\dfrac{n!}{2^{1+2+3+\ldots+n}}}}{2^{\frac{n+1}{2}}\cdot\dfrac{n+1}{2^{n+1}}} = \sqrt[n]{\dfrac{n!}{(n+1)^n}}$

$$\lim_{n\to\infty}\sqrt[n]{\frac{n!}{(n+1)^n}} \overset{D'Alambert}{=} \lim_{n\to\infty}\frac{(n+1)!}{(n+2)^{n+1}}\cdot\frac{(n+1)^n}{n!} = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{n+1} = \frac{1}{e}$$