OLIMPIADA NAȚIONALĂ DE MATEMATICĂ
ETAPA LOCALĂ - BRĂILA, 22 februarie 2015
CLASA A IX-A

1. Fie triunghiul ABC înscris într-un cerc de centru O. Fie P şi Q simetricele ortocentrului şi a vârfului A faţă de mijlocul lui (BC). Demonstraţi că $\overrightarrow{OQ}=\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OP}$

Gazeta Matematică

2. Fie $n \in \mathbb{N}^*$. Aflaţi $x \in \mathbb{Z}$ pentru care $\dfrac{x^{5n}+2x^{2n}-x^n+1}{x^{5n}+x^n-1} \in \mathbb{Z}$.

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3. Fie şirul $(x_n)_{n\geq 1}$, $x_1=\dfrac{1}{2014}$, $x_{n+1}=x_n\left(1+x_1+x_1^2+...+x_1^n\right)$, oricare ar fi $n \geq 1$.

Notăm cu $S=\dfrac{x_1}{x_2}+\dfrac{x_2}{x_3}+...+\dfrac{x_{2014}}{x_{2015}}$. Aflaţi $[S]$.

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4. Fie triunghiul oarecare ABC. Considerăm dreptele $d_1 \| AB$, $d_2 \| AC$, $d_3 \| BC$ şi notăm $d_1 \cap (AC)=\{I\}$, $d_1 \cap (BC)=\{F\}$, $d_2 \cap (AB)=\{D\}$, $d_2 \cap (BC)=\{G\}$, $d_3 \cap (AB)=\{E\}$, $d_3 \cap (AC)=\{H\}$, $d_1 \cap d_2=\{A'\}$, $d_1 \cap d_3=\{B'\}$, $d_2 \cap d_3=\{C'\}$. Presupunem că $\text{Aria}(AEH)=\text{Aria}(CIF)=\text{Aria}(BDG)=\text{Aria}(ADGC)=\text{Aria}(BEHC)=\text{Aria}(AIFB)$.

Aflaţi $\dfrac{Aria(A'B'C')}{Aria(ABC)}$.

Elev Paul Crestez, Brăila

OLIMPIADA NAȚIONALĂ DE MATEMATICĂ
ETAPA LOCALĂ - BRĂILA, 22 februarie 2015
CLASA A IX-A - Soluții

1. Fie triunghiul ABC înscris într-un cerc de centru O. Fie P şi Q simetricele ortocentrului şi a vârfului A faţă de mijlocul lui (BC). Demonstraţi că $\overrightarrow{OQ}=\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OP}$

Gazeta Matematică

Soluție. Fie M mijlocul lui (BC) şi H ortocentrul triunghiului ABC, atunci avem

$$\left.\begin{aligned}2\overrightarrow{OM}&=\overrightarrow{OP}+\overrightarrow{OH}\\2\overrightarrow{OM}&=\overrightarrow{OA}+\overrightarrow{OQ}\end{aligned}\right\}\Rightarrow \overrightarrow{OA}+\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{OH}\Leftrightarrow \overrightarrow{OA}+\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\Rightarrow \overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{OB}+\overrightarrow{OC}$$

2. Fie $n\in\mathbb{N}^*$. Aflaţi $x\in\mathbb{Z}$ pentru care $\dfrac{x^{5n}+2x^{2n}-x^n+1}{x^{5n}+x^n-1}\in\mathbb{Z}$.

Carmen şi Viorel Botea, Brăila

Soluţie. Notăm $x^n=y\Rightarrow$

$$\Rightarrow\frac{y^5+2y^2-y+1}{y^5+y-1}=\frac{\left(y^5+y^2\right)+\left(y^2-y+1\right)}{\left(y^5+y^2\right)-\left(y^2-y+1\right)}=\frac{y^2(y+1)\left(y^2-y+1\right)+\left(y^2-y+1\right)}{y^2(y+1)\left(y^2-y+1\right)-\left(y^2-y+1\right)}=\frac{y^3+y^2+1}{y^3+y^2-1}\in\mathbb{Z}\Rightarrow$$

$$\Rightarrow\begin{cases}y^3+y^2-1\,\big|\,y^3+y^2+1\\ y^3+y^2-1\,\big|\,y^3+y^2-1\end{cases}\Rightarrow \underbrace{y^3+y^2}_{par}-1\,\Big|\,2\Rightarrow y^3+y^2-1=\pm1\Rightarrow$$

$\Rightarrow$ i) $y^3+y^2-1=-1\Rightarrow y=0, y=-1\Rightarrow x=0, x=-1, n$ impar

ii) $y^3+y^2-1=1\Rightarrow y=1\Rightarrow x=\pm1, n$ par şi $x=1, n$ impar,

deci $x\in\{+1,0,1\},(\forall n)\in\mathbb{N}.$

3. Fie şirul $(x_n)_{n\geq1}$, $x_1=\dfrac{1}{2014}$, $x_{n+1}=x_n\left(1+x_1+x_1^2+...+x_1^n\right)$, oricare ar fi $n\geq1$. Notăm cu $S=\dfrac{x_1}{x_2}+\dfrac{x_2}{x_3}+...+\dfrac{x_{2014}}{x_{2015}}$. Aflaţi $[S]$.

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Soluţie. $\dfrac{x_n}{x_{n+1}}=\dfrac{1}{1+x_1+x_1^2+...+x_1^n},(\forall)n\geq 1$.

$$S=\frac{1}{1+\frac{1}{2014}}+\frac{1}{1+\frac{1}{2014}+\left(\frac{1}{2014}\right)^2}+...+\frac{1}{1+\frac{1}{2014}+\left(\frac{1}{2014}\right)^2+...+\left(\frac{1}{2014}\right)^{2014}}<2014\frac{1}{1+\frac{1}{2014}}=$$

$$=\frac{2014^2}{2015}<2014\Rightarrow S<2014$$

$$S>\frac{2014}{1+\frac{1}{2014}+\left(\frac{1}{2014}\right)^2+...+\left(\frac{1}{2014}\right)^{2014}};1+\frac{1}{2014}+\left(\frac{1}{2014}\right)^2+...+\left(\frac{1}{2014}\right)^{2014}=\frac{1-\left(\frac{1}{2014}\right)^{2015}}{1-\frac{1}{2014}}<$$

$$<\frac{2014}{2013}\Rightarrow S>\frac{2014\cdot 2013}{2014}=2013\Rightarrow[S]=2013$$

4. Fie triunghiul oarecare ABC. Considerăm dreptele $d_1\|AB,\ d_2\|AC,\ d_3\|BC$ şi notăm $d_1\cap(AC)=\{I\},\ d_1\cap(BC)=\{F\},\ d_2\cap(AB)=\{D\},\ d_2\cap(BC)=\{G\},\ d_3\cap(AB)=\{E\},$ $d_3\cap(AC)=\{H\},\ d_1\cap d_2=\{A'\},\ d_1\cap d_3=\{B'\},\ d_2\cap d_3=\{C'\}$. Presupunem că $\text{Aria}(AEH)=\text{Aria}(CIF)=\text{Aria}(BDG)=\text{Aria}(ADGC)=\text{Aria}(BEHC)=\text{Aria}(AIFB)$.

Aflați $\dfrac{Aria(A'B'C')}{Aria(ABC)}$.

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Soluţie. Deoarece

$$EH\|BC\Rightarrow\triangle AEH\sim\triangle ABC\Rightarrow\left(\frac{AE}{AB}\right)^2=\frac{Aria(\triangle AEH)}{Aria(\triangle ABC)}=\frac{Aria(\triangle AEH)}{Aria(\triangle AEH)+Aria(\triangle EBCH)}=$$

$$=\frac{Aria(\triangle AEH)}{2Aria(\triangle AEH)}=\frac{1}{2}\Rightarrow\frac{AE}{AB}=\frac{1}{\sqrt{2}}\Rightarrow AE=\frac{AB}{\sqrt{2}}\Rightarrow EB=AB-AE=AB-\frac{AB}{\sqrt{2}}=AB\frac{\sqrt{2}-1}{\sqrt{2}}.$$

$EH\|BC$ şi $AB\|IF\Rightarrow EB'FB$ paralelogram $\Rightarrow B'F=EB=AB\dfrac{\sqrt{2}-1}{\sqrt{2}},(1)$. Deoarece $DG\|AC\Rightarrow$

$$\Rightarrow\triangle BDG\sim\triangle BAC\Rightarrow AD=AB\frac{\sqrt{2}-1}{\sqrt{2}}.$$

$IF\|AB$ şi $DG\|AC\Rightarrow AIA'D$ paralelogram $\Rightarrow A'I=AD=AB\dfrac{\sqrt{2}-1}{\sqrt{2}},(2)$.

$$IF \parallel AB \Rightarrow \triangle CIF \sim \triangle CAB \Rightarrow \left(\frac{IF}{AB}\right)^2 = \frac{Aria(\triangle CIF)}{Aria(\triangle CAB)} = \frac{Aria(\triangle CIF)}{Aria(\triangle CIF) + Aria(\triangle IFBA)} = \frac{Aria(\triangle CIF)}{2Aria(\triangle CIF)} =$$

$= \frac{1}{2} \Rightarrow \frac{IF}{AB} = \frac{1}{\sqrt{2}} \Rightarrow IF = \frac{AB}{\sqrt{2}}, (3)$. Din $(1),(2)(3)$ avem $A'B' = IF - IA' - IB' = \frac{AB}{\sqrt{2}} - AB\frac{\sqrt{2}-1}{\sqrt{2}} -$

$$-AB\frac{\sqrt{2}-1}{\sqrt{2}} = AB\frac{3-2\sqrt{2}}{\sqrt{2}}.$$

$$\left.\begin{array}{r} A'B' \parallel AB \\ A'C' \parallel AC \\ B'C' \parallel BC \end{array}\right\} \Rightarrow \triangle ABC \sim \triangle A'B'C' \Rightarrow \frac{Aria(\triangle A'B'C')}{Aria(\triangle ABC)} = \left(\frac{A'B'}{AB}\right)^2 \Rightarrow \frac{Aria(\triangle A'B'C')}{Aria(\triangle ABC)} = \left(\frac{3-2\sqrt{2}}{\sqrt{2}}\right)^2 \Rightarrow$$

$$\Rightarrow \frac{Aria(\triangle A'B'C')}{Aria(\triangle ABC)} = \frac{17-12\sqrt{2}}{2}.$$