BAREM clasa a XI a

21 februarie 2016

Subiectul I

Fie $A=\left(a_{ij}\right)_{1\le i,j\le n}$ și $S_1=\sum_{i=1}^{n}a_{1i}$, $S_2=\sum_{i=1}^{n}a_{2i}$, ..., $S_n=\sum_{i=1}^{n}a_{ni}$, $S_1'=\sum_{i=1}^{n}a_{i1}$, $S_2'=\sum_{i=1}^{n}a_{i2}$,..., $S_n'=\sum_{i=1}^{n}a_{in}$..2p

Cum $S_1+S_2+\dots+S_n=S_1'+S_2'+\dots+S_n'$ și $S_1+S_2+\cdots S_n\ge 0$, iar $S_1'+S_2'+\dots+S_n'\le 0$

$\Rightarrow S_1+S_2+\dots+S_n=0=S_1'+S_2'+\dots+S_n'$

Cum $S_k\ge 0,\forall k=\overline{1,n}\Longrightarrow S_k=0$. Analog $S_k'=0$...3p

Deci $S_1=S_2=\dots=S_n=0$.

Atunci $detA=0$! (pentru că, de exemplu, se adună la prima coloană restul coloanelor)........2p

Subiectul II

a) $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$; $B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$. Se verifică prin calcul. 1p

b) Din $Tr(AB)=Tr(BA)\Rightarrow Tr(AB-BA)=0$.
Atunci din Teorema Cayley-Hamilton $\Rightarrow (AB-BA)^2+det(AB-BA)I_2=O_2$
$\Longrightarrow -(AB-BA)^2=det(AB-BA)I_2$...2p
Atunci $det(AB+BA)I_2+det(AB-BA)I_2=\big(det(AB+BA)+det(AB-BA)\big)I_2=$
$2\big(det(AB)+det(BA)\big)I_2=4\,det(AB)I_2$...2p
Atunci $Tr(det(AB+BA)I_2-(AB-BA)^2)=Tr(4\,det(AB)I_2)\,\vdots\,8$.............................2p

Subiectul III

a) Dacă $(a_n)=convergent\ \Rightarrow\ \lim\limits_{n\to\infty}a_n=l\in\mathbb{R}$. Din ipoteză rezultă $l=l^2-8l+18\Rightarrow l_1=3;\ l_2=6\ \Longrightarrow l\in\{3,6\}$..1p
Presupunem prin reducere la absurd că $a_{n+1}\ne a_n\ \forall n\in\mathbb{N}$.
Atunci $a_{n+1}-a_n=a_n^2-8a_n-a_{n-1}^2+8a_{n-1}=(a_n-a_{n-1})(a_n+a_{n-1})-8(a_n-a_{n-1})=$
$=(a_n-a_{n-1})(a_n+a_{n-1}-8)$..1p

$\Rightarrow |a_{n+1}-a_n|=|a_n-a_{n-1}||a_n+a_{n-1}-8|$ $(convergent\ la\ 4\ pt\ a_n\to 6\ sau\ la\ 2\ pt\ a_n\to 3)\Longrightarrow \exists n_0\in\mathbb{N}\ ai\ \forall n>n_0\ avem\ |a_n+a_{n-1}-8|>1{,}5$..1p

Atunci $|a_{n+1}-a_n| = |a_n-a_{n-1}||a_n+a_{n-1}-8| > 1{,}5|a_n-a_{n-1}| > 1{,}5^2|a_{n-1}-a_{n-2}| >$
$\ldots > 1{,}5^{n-n_0}|a_{n0+1}-a_{n0}|$..1p

Deci $|a_{n+1}-a_n| > 1{,}5^{n-n0}|a_{n0+1}-a_{n0}|\ \forall n > n_0$.

Prin trecere la limită $\Rightarrow 0 > \infty$ fals. $\exists p \in \mathbb{N}\ ai\ a_{p+1} = a_p \Rightarrow a_{p+1}^2 - 8a_{p+1} + 18 = a_p^2 - 8a_p + 18 \Rightarrow a_{p+2} = a_{p+1} \Rightarrow a_{n+1} = a_n\ \forall n \geq p$ (prin inducție)..1p

b) Dacă $a_n = convergent \Rightarrow conform\ a)\ \exists p \in \mathbb{N}\ ai\ a_{n+1} = a_n\ \forall n \geq p$.
Dacă $a_p = k \Rightarrow a_n = k\ \forall n \geq p \Rightarrow a_n \to k \Rightarrow k \in \{3, 6\}$..1p

1) $k = 3 \Rightarrow a_p = 3 \Rightarrow a_{p-1}^2 - 8a_{p-1} + 18 = 3 \Rightarrow a_{p-1}^2 - 8a_{p-1} + 15 = 0$

$\Delta = 4 \Rightarrow a_{p-1} \in$ dacă $a_{p-1} = 5 \Rightarrow a_{p-2}^2 - 8a_{p-2} + 18 = 5 \Rightarrow a_{p-2}^2 - 8a_{p-2} + 13 = 0$

$cu\ \Delta = 12a_{p-2} \in \mathbb{R}\backslash\mathbb{Q}$

2) $k = 6 \Rightarrow a_p = 6 \Rightarrow a_{p-1}^2 - 8a_{p-1} + 18 = 6 \Rightarrow a_{p-1}^2 - 8a_{p-1} + 12 = 0$
$\Delta = 64 - 48 = 16 \Rightarrow a_{p-1} \in \{2, 6\}$
$a_{p-1} = 2 \Rightarrow a_{p-2}^2 - 8a_{p-2} + 16 = 0 \Rightarrow a_{p-2} = 4 \Rightarrow a_{p-3}^2 - 8a_{p-3} + 14 = 0$
$\Rightarrow \Delta = 8 \Rightarrow a_{p-3} \notin \mathbb{Q}$
Plecăm de la $\{2, 3, 4, 5\ sau\ 6\} \Rightarrow deci\ se\ poate\ ajunge\ la\ a_p \in \{3, 6\}$............................1p

Subiectul I

Obs. că $x_n = \prod_{k=1}^{n}\left(\sqrt[k+1]{1+\frac{1}{k}} - 1\right) > 0$ 1p

Avem $\sqrt[k+1]{\left(1+\frac{1}{k}\right)\cdot \underbrace{1\cdot 1\cdot \ldots \cdot 1}_{k\ ori}} < \frac{1+\frac{1}{k}+k}{k+1} = \frac{k^2+k+1}{k(k+1)} \Rightarrow \sqrt[k+1]{1+\frac{1}{k}} - 1 < \frac{1}{k(k+1)} \Rightarrow$ 3p

$\prod_{k=1}^{n}\left(\sqrt[k+1]{1+\frac{1}{k}} - 1\right) < \prod_{k=1}^{n}\frac{1}{k(k+1)} = \frac{1}{1\cdot 2}\cdot\frac{1}{2\cdot 3}\cdot\frac{1}{3\cdot 4}\cdot \ldots \cdot \frac{1}{n(n+1)} = \frac{1}{(n!)^2(n+1)}$ 2p

Deci avem inegalitatea $0 < x_n < \frac{1}{(n!)^2(n+1)} \Rightarrow \lim_{n\to\infty} x_n = 0$. 1p