Barem clasa a XI-a
(OLM 2016-etapa locală)

Subiectul I. (7 puncte)

$a)\ detA = 5 \neq 0 \to \exists A^{-1} \to A^2 - 3A = -5I_2 \to A(A-3I_2) = -5I_2 \to A^{-1} = -\frac{1}{5}(A-3I_2)$ **(4 puncte)**

$b) \begin{cases} A^2 - I_2 = (A-I_2)(A+I_2) \\ A^2 + A = A(A+I_2) \\ A^2 + 2I_2 = 3A - 3I_2 = 3(A-I_2) \end{cases}$ și $f(x) = \det(A - xI_2) = x^2 - 3x + 5$. Atunci **(1 punct)**

$$\begin{cases} det(A^2 - I_2) = det\left((A-I_2)(A+I_2)\right) = f(1)\cdot f(-1) = 3\cdot 9 = 27 \\ det(A^2 + A) = det\left(A(A+I_2)\right) = 5\cdot f(-1) = 5\cdot 9 = 45 \\ det(A^2 + 2I_2) = det(3(A-I_2)) = 9\cdot f(1) = 9\cdot 3 = 27 \end{cases}$$ **(1 punct)**

$det(A^2 - I_2) + det(A^2 + A) - det(A^2 + 2I_2) = 45$ **(1 punct)**

Subiectul II. (7 puncte)

Relația de recurență se poate scrie sub forma $\frac{1}{x_n} = an + \frac{1}{x_{n-1}}, n \geq 2$ **(2 puncte)**

Notăm $y_n = \frac{1}{x_n}$, obținem $y_n = y_{n-1} + an$ și deci $y_n = \frac{a}{2}n(n+1), \forall n \in N$ **(2 puncte)**

$x_n = \frac{1}{y_n} = \frac{2}{a}\left(\frac{1}{n} - \frac{1}{n+1}\right), \forall n \in N \Rightarrow x_1 + x_2 + ... + x_n = \frac{2}{a}\frac{n}{n+1}, \forall n \in N$ **(2 puncte)**

$\lim\limits_{n\to\infty}(x_1 + x_2 + ... + x_n) = \frac{2}{a}$ **(1 punct)**

Subiectul III. (7 puncte)

In relaţia de recurenţă, împărţind cu $(n+1)!$, avem $\frac{x_{n+1}}{(n+1)!} - \frac{x_n}{n!} = \frac{n+1}{3^{n+1}}$. **(1 punct)**

Dând valori lui n, obţinem că $\frac{x_n}{n!} - \frac{x_1}{1!} = \frac{2}{3^2} + \ldots + \frac{n}{3^n}$, deci $\frac{x_n}{n!} = \frac{1}{3} + \frac{2}{3^2} + \ldots + \frac{n}{3^n}$. **(2 puncte)**

Se arata că $\frac{1}{3} + \frac{2}{3^2} + \ldots + \frac{n}{3^n} = \frac{3}{4}\left(1 - \left(\frac{1}{3}\right)^n\right) - \frac{3}{2}\cdot\frac{n}{3^{n+1}}$, de unde $\lim\limits_{n\to\infty}\frac{x_n}{n!} = \frac{3}{4}$. **(2 puncte)**

Putem scrie $\lim\limits_{n\to\infty} n\left(\frac{x_n}{n!} - \frac{3}{4}\right) = \lim\limits_{n\to\infty}\frac{\frac{x_n}{n!} - \frac{3}{4}}{\frac{1}{n}} \overset{Stolz\ Cesaro}{=} \lim\limits_{n\to\infty}\frac{\frac{x_{n+1}}{(n+1)!} - \frac{x_n}{n!}}{\frac{1}{n+1} - \frac{1}{n}} = \lim\limits_{n\to\infty}\frac{\frac{n+1}{3^{n+1}}}{-\frac{1}{n(n+1)}} = -\lim\limits_{n\to\infty}\frac{n(n+1)^2}{3^{n+1}} = 0$. **(2 puncte)**

Subiectul IV. (7 puncte)

a) Fie $E(n) = \left(\frac{n^2+3}{n^2+2}\right)\left(\frac{n^2+5}{n^2+4}\right)\cdot\ldots\cdot\left(\frac{n^2+2n+2017}{n^2+2n+2016}\right) = \left(1 + \frac{1}{n^2+2}\right)\cdot\left(1 + \frac{1}{n^2+4}\right)\cdot\ldots\cdot\left(1 + \frac{1}{n^2+2n+2016}\right)$. **(1 punct)**

Aplicăm criteriul cleştelui pentru a calcula limita

$$\left(1 + \frac{1}{n^2+2n+2016}\right)^{n+1008} \leq \left(1 + \frac{1}{n^2+2}\right)\cdot\left(1 + \frac{1}{n^2+4}\right)\cdot\ldots\cdot\left(1 + \frac{1}{n^2+2n+2016}\right) \leq \left(1 + \frac{1}{n^2+2}\right)^{n+1008}$$

Deoarece $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n^2+2}\right)^{n+1008} = \lim\limits_{n\to\infty}\left(1 + \frac{1}{n^2+2n+2016}\right)^{n+1008} = e^0 = 1$, *conform criteriului cleştelui* **(1 punct)**

$\lim\limits_{n\to\infty}\frac{(n^2+3)\cdot(n^2+5)\cdot\ldots\cdot\left[(n+1)^2+2016\right]}{(n^2+2)\cdot(n^2+4)\cdot\ldots\cdot\left[(n+1)^2+2015\right]} = 1$ **(1 punct)** *b)* *Observăm nedeteminarea* 1^{∞} *,* $l=\lim\limits_{n\to\infty}\left(1+\frac{\sqrt[n]{4}+\sqrt[n]{504}-2}{2}\right)^{2n}$, $l=e^{\lim\limits_{n\to\infty}2n\left(\frac{4^{\frac{1}{n}}-1+504^{\frac{1}{n}}-1}{2}\right)}$ **(2 puncte)**

Aplicăm limita remarcabilă $\lim\limits_{n\to\infty}\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}=\ln a, a\succ 0,$ *de unde obţinem:* $l=e^{\ln 4+\ln 504}=e^{\ln 2016}=2016$. **(2 puncte)**