

Călărași 2015

Problema 1. Fie $ABCD$ un patrulater inscriptibil, fie O punctul de intersecție a diagonalelor AC și BD , fie I centrul cercului înscris în triunghiul ABC și fie J centrul cercului înscris în triunghiul ABD . Dreapta IJ intersectează segmentul OA , respectiv OB , în punctul M , respectiv N . Arătați că triunghiul OMN este isoscel.

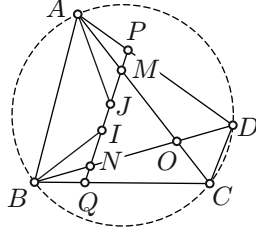
Problema 2. Arătați că muchiile unui graf finit simplu (fără bucle și fără muchii multiple) conex pot fi orientate astfel încât numărul de săgeți care ies din fiecare vârf să fie par, dacă și numai dacă numărul total de muchii este par.

Problema 3. Determinați numerele naturale nenule n , care au următoarea proprietate: toate numerele naturale mai mici decât n , relativ prime cu n , sunt relativ prime două câte două.

Problema 4. Fie n un număr natural mai mare sau egal cu 2. Determinați numerele care pot fi scrise sub forma $\sum_{i=2}^k a_{i-1}a_i$, unde k este un număr natural mai mare sau egal cu 2, iar a_1, \dots, a_k sunt numere naturale nenule, a căror sumă este n .

Călărași 2015 — Solutions

Problem 1. Let $ABCD$ be a cyclic quadrangle, let the diagonals AC and BD cross at O , and let I and J be the incentres of the triangles ABC and ABD , respectively. The line IJ crosses the segments OA and OB at M and N , respectively. Prove that the triangle OMN is isosceles.



Solution. We show that $\angle OMN \equiv \angle ONM$. To this end, let the line IJ cross the segments AD and BC at P and Q , respectively, and consider the position of J relative to the line AC , to write $\angle OMN \equiv \angle AJP \pm \angle JAM \equiv \angle AJP \pm (\pm \angle JAD \mp \angle CAD) \equiv \angle AJP + \angle JAD - \angle CAD \equiv \angle AJP + \angle BAJ - \angle CAD$. Similarly, $\angle ONM \equiv \angle BIQ + \angle ABI - \angle CBD$. Since the quadrangle $ABCD$ is cyclic, $\angle CAD \equiv \angle CBD$ and $\angle ACB \equiv \angle ADB$. The latter implies that $\angle AIB \equiv \angle AJB$, so the quadrangle $ABIJ$ is cyclic. Consequently, $\angle AJP \equiv \angle ABI$ and $\angle BAJ \equiv \angle BIQ$, and the conclusion follows.

Problem 2. Show that the edges of a connected simple (no loops and no multiple edges) finite graph can be oriented so that the number of edges leaving each vertex is even if and only if the total number of edges is even.

Solution 1. Given any orientation, the total number of edges equals the sum of all out-degrees. If the latter are all even, then so is the former.

To establish the converse, induct on the number of edges to show that any connected simple finite graph with an even number of edges splits into edge-disjoint paths of length 2. Orient the two edges of each of these paths away from the joint to obtain the required orientation.

Solution 2. The converse can alternatively be established as follows. Given any orientation, the total number of edges and the number of vertices that have an odd out-degree have the same parity. If the former is even, then so is the latter. Consider an orientation producing a minimal number of vertices of odd out-degree. If this number is positive, choose two such vertices and use connectedness to join them by a path. Reverting orientations along the path changes the parity of out-degrees only at the end-points, so the outcome is an oriented graph with fewer vertices of odd out-degree. This contradicts minimality and concludes the proof.

Problem 3. Determine all positive integers n such that all positive integers less than n and prime to n are pairwise coprime.

Solution. Let $p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots$ be the sequence of primes and let p_m be the first prime which does not divide n . A necessary and sufficient condition that n be of the required type is that $n < p_m^2$. Since each of the primes p_1, \dots, p_{m-1} divides n , so does their product, hence $p_1 \cdots p_{m-1} \leq n < p_m^2$.

Notice that 5 is the first index k such that $p_1 \cdots p_{k-1} > p_k^2$. Now, if $p_1 \cdots p_{k-1} > p_k^2$ for some index $k \geq 5$, then $p_1 \cdots p_{k-1} p_k > p_k^3 > 4p_k^2 > p_{k+1}^2$, by the Bertrand-Tchebysheff theorem, so $p_1 \cdots p_{k-1} > p_k^2$ for all indices $k \geq 5$.

Consequently, $m \leq 4$, so $n < p_4^2 = 49$. Examination of the possible cases quickly yields the required numbers: 1, 2, 3, 4, 6, 8, 12, 18, 24, 30.

Problem 4. Given an integer $n \geq 2$, determine the numbers that can be written in the form $\sum_{i=2}^k a_{i-1}a_i$, where k is an integer greater than or equal to 2, and a_1, \dots, a_k are positive integers that add up to n .

Solution. The required numbers are the integers in the range $n - 1, \dots, \lfloor n^2/4 \rfloor$.

Given an integer $k \geq 2$ and positive integers a_1, \dots, a_k that add up to n , we show that $n - 1 \leq \sum_{i=2}^k a_{i-1}a_i \leq \lfloor n^2/4 \rfloor$. To prove the first inequality, write

$$\sum_{i=2}^k a_{i-1}a_i = a_1a_2 + \sum_{i=3}^k a_{i-1}a_i \geq a_1 + a_2 - 1 + \sum_{i=3}^k a_{i-1}a_i \geq a_1 + a_2 - 1 + \sum_{i=3}^k a_i = n - 1.$$

To prove the second, write

$$\begin{aligned} \sum_{i=2}^k a_{i-1}a_i &\leq \left(\sum_{i=1}^{\lfloor (k+1)/2 \rfloor} a_{2i-1} \right) \left(\sum_{i=1}^{\lfloor k/2 \rfloor} a_{2i} \right) = \left(\sum_{i=1}^{\lfloor (k+1)/2 \rfloor} a_{2i-1} \right) \left(\sum_{i=1}^k a_i - \sum_{i=1}^{\lfloor (k+1)/2 \rfloor} a_{2i-1} \right) \\ &\leq \frac{1}{4} \left(\sum_{i=1}^k a_i \right)^2 = \frac{n^2}{4}. \end{aligned}$$

Since $\sum_{i=2}^k a_{i-1}a_i$ is integral, it cannot exceed $\lfloor n^2/4 \rfloor$.

Conversely, let s be an integer in the range $n - 1, \dots, \lfloor n^2/4 \rfloor$. If s is the product of two positive integers that add up to n , we are done; in particular, this is the case for $n - 1 = 1 \cdot (n - 1)$ and $\lfloor n^2/4 \rfloor = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$. Otherwise, consider the largest integer $m < \lfloor n/2 \rfloor$ such that $m(n - m) < s$, to infer that $\ell = s - m(n - m)$ is a positive integer less than $n - 2m - 1$. Letting $a_1 = n - 2m - \ell - 1$, $a_2 = m$, $a_3 = \ell + m$, and $a_4 = 1$, it is readily checked that $a_1 + a_2 + a_3 + a_4 = n$, and $s = a_1a_2 + a_2a_3 + a_3a_4$.